Rotational stagnation point flow

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The constant-density inviscid rotational flow in the neighbourhood of a general stagnation point on a wall is investigated. In all but very special cases, the solution is non-analytic and the vorticity at the wall is infinite; the stagnation streamline is tangent to the wall at the stagnation point; stagnation points of saddle-point type cannot exist.

The boundary-layer equations corresponding to the inviscid solutions studied are presented.

1. Introduction

It is well known that the splitting streamline at a stagnation point in a planar rotational flow makes a finite angle with the normal to the body. The present paper was suggested by the question of what the analogous result would be at a general stagnation point in a three-dimensional flow. The answer to this question turns out to be that the stagnation streamline in this case comes in tangent to the body in general.

An analysis is made of the constant-density flow on one side of a plane wall in which the lateral velocity components are linear functions of the lateral variables and the normal component is independent of the lateral variables. All components are general functions of distance from the wall. Such a flow can simulate in local behaviour a general compressible flow near a general stagnation point on a body of finite curvature. A constant viscosity coefficient is introduced into the general equations for the purpose of permitting the study of boundary layers in such flows. However, our primary interest is focused on the inviscid equations, and particularly on their non-analytic solutions.

The normal component of vorticity at a general inviscid stagnation point is necessarily zero. This will be shown in our case but can easily seem to be true at the wall in much greater generality. The general stagnation point is characterized by a basic parameter α_0. If $\alpha_0^2 < 1$ the streamline pattern on the wall is a node, and this will be the range of our primary interest. The case $\alpha_0 = 0$ represents axisymmetric or almost-axisymmetric flow, with the wall streamline pattern a source. The case $\alpha_0^2 = 1$ represents planar or almost-planar flow, with a degenerate wall streamline pattern. The case $\alpha_0^2 > 1$ represents flows for which the wall streamline pattern is a saddle-point. The solution types arising in these various cases are naturally quite different in nature.

The effect to be investigated may be described in physical terms as follows: There is a distribution of lateral vorticity in the flow approaching the wall.

This vorticity becomes amplified by the lateral stretching of vortex lines in the stagnation region flow. The lateral vorticity approaches infinity at the wall. It is the resulting singular behaviour in the velocity field which is of interest to us.

2. Basic equations

The space of interest is the part of physical space with Cartesian co-ordinates (ax, ay, az) for which $z \geqslant 0$. The wall is the plane $z = 0$. The quantity a is a reference length which may be chosen arbitrarily. The normal flow approaching the wall is characterized by a reference velocity gradient U'. The velocity vector in matrix notation is assumed to have the solenoidal form

$$[\mathbf{q}] = \tfrac{1}{2}U'a[F+x(H'-\alpha)+y(\beta-\gamma), G+x(\beta+\gamma)+y(H'+\alpha), -2H]. \tag{2.1}$$

Here F, G, H, α, β, and γ are functions of z alone. Primes will denote differentiation, except in the symbol U'. The vorticity corresponding to (2.1) may be expressed as

$$[\nabla\times\mathbf{q}] = \tfrac{1}{2}U'[-G'-x(\beta'+\gamma')-y(H''+\alpha), F'+x(H''-\alpha')+y(\beta'-\gamma'), 2\gamma]. \tag{2.2}$$

The momentum equation

$$\mathbf{q}.\nabla\mathbf{q}+\rho^{-1}\nabla p = \nu\nabla^2\mathbf{q} \tag{2.3}$$

yields for the pressure gradient

$$[\nabla p] = \tfrac{1}{2}\rho U'^2a[M+x(D-A)+y(B-C), N+x(B+C)+y(D+A), -2HH'-2R^{-1}H''], \tag{2.4}$$

where

$$A = H\alpha'-H'\alpha+R^{-1}\alpha'', \tag{2.5a}$$
$$B = H\beta'-H'\beta+R^{-1}\beta'', \tag{2.5b}$$
$$C = H\gamma'-H'\gamma+R^{-1}\gamma'', \tag{2.5c}$$
$$D = HH''-H'^2+\tfrac{1}{2}(H'^2-\alpha^2-\beta^2+\gamma^2)+R^{-1}H''', \tag{2.5d}$$
$$M = HF'-\tfrac{1}{2}(H'-\alpha)F-\tfrac{1}{2}(\beta-\gamma)G+R^{-1}F'', \tag{2.5e}$$
$$N = HG'-\tfrac{1}{2}(H'+\alpha)G-\tfrac{1}{2}(\beta+\gamma)F+R^{-1}G'', \tag{2.5f}$$

and

$$R = U'a^2/\nu. \tag{2.5g}$$

The curl of the pressure gradient is then

$$[\nabla\times(\nabla p)] = \tfrac{1}{2}\rho U'^2[-N'-x(B'+C)'-y(D'+A'), M'+x(D'-A')+y(B'-C'), 2C]. \tag{2.6}$$

The vanishing of this vector is the condition of integrability for the pressure, and leads to the results

$$C = 0, \quad A, B, D, M, N = \text{const.} \tag{2.7}$$

The pressure may now be expressed as

$$p = p_{st}+\tfrac{1}{2}\rho U'^2a^2\{\tfrac{1}{2}D(x^2+y^2)+\tfrac{1}{2}A(y^2-x^2)+Bxy+Mx+Ny-H^2-2R^{-1}H'\}. \tag{2.8}$$

The constant B can be eliminated by a rotation of axes; thus, without loss of generality we can choose the x- and y-axes as principal axes and set

$$B = 0. \tag{2.9}$$

Analogously, by a suitable choice of the location of the origin on the plane $z = 0$ we can set

$$M = 0 \tag{2.10}$$

as long as $D-A \neq 0$, and can set

$$N = 0 \tag{2.11}$$

as long as $D+A \neq 0$. Equations (2.5) are now the equations governing the flow. An important feature is that the first four equations may be solved without the last two, and that the last two are then strictly linear in F and G. We shall term the solution of the first four equations the *primary* solution and that of the last two as the *secondary* solution.

In inviscid flow we set

$$R^{-1} = 0 \tag{2.12}$$

and impose the boundary conditions

$$H(0) = 0, \quad H'(0) = 1, \quad \alpha(0) = \alpha_0. \tag{2.13a, b, c}$$

The second of these corresponds to the identification of the physical normal gradient of normal velocity with the quantity U'. With these boundary conditions we have

$$\beta(0) = 0, \quad \gamma(0) = 0, \quad F(0) = 0, \quad G(0) = 0, \tag{2.14a, b, c, d}$$

provided $M = 0$ and $N = 0$, and

$$A = -\alpha_0 = -\alpha(0), \quad D = -\tfrac{1}{2}(1+\alpha_0^2). \tag{2.15a, b}$$

We shall use these expressions for A and D for viscous flow also, without the identification of α_0 as $\alpha(0)$ and of 1 as $H'(0)$. Without loss of generality, we can restrict α_0 to non-negative values. Note that the result (2.14*b*) is independent of choice of axes or origin, and states that the normal component of vorticity is zero everywhere on the wall.

In the inviscid case, the integrals for β and γ may be obtained immediately. They are

$$\beta = \beta_0' H, \quad \gamma = \gamma_0' H. \tag{2.16a, b}$$

The basic equations then become

$$H\alpha' + \alpha_0 - H'\alpha = 0, \tag{2.17a}$$

$$HH'' + \tfrac{1}{2}(1+\alpha_0^2 - H'^2 - \alpha^2) - \tfrac{1}{2}(\beta_0'^2 - \gamma_0'^2) H^2 = 0, \tag{2.17b}$$

$$HF' - \tfrac{1}{2}(H'-\alpha) F - \tfrac{1}{2}(\beta_0' - \gamma_0') HG = 0, \tag{2.18a}$$

$$HG' - \tfrac{1}{2}(H'+\alpha) G - \tfrac{1}{2}(\beta_0' + \gamma_0') HF = 0, \tag{2.18b}$$

provided $M = 0$ and $N = 0$.

It is clear that H is to be interpreted as normal velocity and γ as normal vorticity. The quantity α is to be interpreted as measuring the distortion of the flow field from axial symmetry, or alternatively $H'-\alpha$ and $H'+\alpha$ as the x and y components, respectively, of the basic flow out from the stagnation point. The quantity β essentially measures the twist of the principal axes away from the wall.

3. The inviscid case $\alpha_0 = 0$

If $\alpha_0 = 0$, we can write the solution for α immediately as

$$\alpha = \alpha_0' H \tag{3.1}$$

in analogy with (2.16). Equation (2.17*b*) becomes

$$HH'' + \tfrac{1}{2}(1 - H'^2) - \tfrac{1}{2}k^2 H^2 = 0, \tag{3.2}$$

where

$$k^2 = \alpha_0'^2 + \beta_0'^2 - \gamma_0'^2. \tag{3.3}$$

The general solution of (3.2) subject to (2.13) is

$$H = k^{-1} \sinh kz + k^{-2} H_0'' (\cosh kz - 1). \tag{3.4}$$

Thus in this case the principal solution is always analytic. If k^2 in (3.3) is negative the hyperbolic functions in (3.4) are replaced by trigonometric ones.

The secondary solution (the solution of (2.18)) may be expressed in the form

$$F = [f_0 \cosh \tfrac{1}{2}kz + k^{-1}\{(\beta_0' - \gamma_0')\, g_0 - \alpha_0' f_0\} \sinh \tfrac{1}{2}kz]\, H^{\frac{1}{2}}, \tag{3.5a}$$

$$G = [g_0 \cosh \tfrac{1}{2}kz + k^{-1}\{(\beta_0' + \gamma_0')\, f_0 + \alpha_0' g_0\} \sinh \tfrac{1}{2}kz]\, H^{\frac{1}{2}}, \tag{3.5b}$$

where f_0 and g_0 are arbitrary constants. Both F and G have a $z^{\frac{1}{2}}$ non-analyticity near the wall, and the vorticity has a $z^{-\frac{1}{2}}$ behaviour near the wall.

The stagnation streamline has a shape near the stagnation point described by

$$-x/f_0 \doteq -y/g_0 \doteq \tfrac{1}{2}z^{\frac{1}{2}}. \tag{3.6}$$

It is therefore locally parabolic in shape and is tangent to the wall at the stagnation point.

4. The inviscid cases $0 < \alpha_0 < 1$ and $1 < \alpha_0$

An analytic primary solution exists in the case $\alpha_0^2 \neq 1$. This solution is given by

$$\alpha = \alpha_0 H', \quad H = k^{-1} \sinh kz, \tag{4.1a, b}$$

where

$$k^2 = (\beta_0'^2 - \gamma_0'^2)/(1 - \alpha_0^2). \tag{4.2}$$

This solution is rather special, with no free parameters in the solution for α and H. The secondary solution is locally of the form

$$F = \left\{ f_0 z^{\frac{1}{2}(1-\alpha_0)} + \frac{\beta_0' - \gamma_0'}{2(1+\alpha_0)} g_0 z^{\frac{1}{2}(3+\alpha_0)} \right\} \{1 + O(k^2 z^2)\}, \tag{4.3a}$$

$$G = \left\{ g_0 z^{\frac{1}{2}(1+\alpha_0)} + \frac{\beta_0' + \gamma_0'}{2(1-\alpha_0)} f_0 z^{\frac{1}{2}(3-\alpha_0)} \right\} \{1 + O(k^2 z^2)\}, \tag{4.3b}$$

where f_0 and g_0 are arbitrary constants.

Note that if $\alpha_0 > 1$ the solution for F yields infinite velocity at the wall with $f_0 \neq 0$. The physical interpretation of this result is that a stagnation point of the type with $\alpha_0 > 1$ cannot exist with non-zero vorticity on the stagnation streamline (with non-zero f_0). With $0 < \alpha_0 < 1$ and $f_0 \neq 0$ or $g_0 \neq 0$ the vorticity is infinite at the wall. The stagnation streamline near the stagnation point is described by

$$-x/f_0 \doteq z^{\frac{1}{2}(1-\alpha_0)}/2(1-\alpha_0), \quad -y/g_0 \doteq z^{\frac{1}{2}(1+\alpha_0)}/2(1+\alpha_0). \tag{4.4a, b}$$

This result reduces to (3.6) in the case $\alpha_0 = 0$.

With $1 > \alpha_0 > 0$ there is a non-analytic primary solution which is locally of the form

$$H' = 1 + h_0 z^{1-\alpha_0} + h_1 z^{1+\alpha_0} + O(z^{2(1-\alpha_0)}), \qquad (4.5a)$$

$$\alpha = \alpha_0 - h_0 z^{1-\alpha_0} + h_1 z^{1+\alpha_0} + O(z^{2(1-\alpha_0)}), \qquad (4.5b)$$

where h_0 and h_1 are arbitrary constants. With $h_0 \neq 0$ the vorticity is infinite on the wall away from the stagnation point. The secondary solution is of a form similar to that of (4.3), except that the order bracket is

$$\{1 + O(h_0 z^{1-\alpha_0}) + O(h_1 z^{1+\alpha_0})\}.$$

With $1 < \alpha_0$ the solution of (4.5) with $h_0 = 0$ is available, but the degree of freedom in the parameter h_0 is lost. In this case $H' + \alpha - 1 - \alpha_0$ is identically zero. A bounded solution for H' and α does exist for which $H' + \alpha - 1 - \alpha_0$ is not identically zero. However this solution corresponds to $H'(0) = \alpha_0$ and $\alpha(0) = 1$, in violation of the conditions (2.13*b*, *c*); in this case the basic parameter α_0 is essentially replaced by a_0^{-1}. This solution is rejected.

5. The inviscid case $\alpha_0 = 1$

If $\beta_0'^2 = \gamma_0'^2$ a primary analytic solution is available, and is given by

$$H = k^{-1} \sinh kz, \quad \alpha = H' = \cosh kz, \qquad (5.1a, b)$$

where k is arbitrary. In this case M cannot generally be set equal to zero, and F and G are given by

$$F = M \ln \frac{\sinh kz}{1 + \cosh kz} + f_0, \quad G = g_0 H. \qquad (5.2a, b)$$

Unless both f_0 and M are zero no stagnation point appears in this case.

The stagnation streamline shape (with $M = 0$ and $f_0 = 0$) near the stagnation point is described by

$$x = 0, \quad y \doteq -\tfrac{1}{4} g_0 z. \qquad (5.3a, b)$$

The result is the well-known one mentioned at the beginning of the introduction, with the stagnation streamline coming into the wall at a finite angle to the normal.

To investigate the more general solutions let us introduce new variables by

$$\theta = \tfrac{1}{2}(H' + \alpha) - 1, \quad \phi = \tfrac{1}{2}(H' - \alpha), \quad H = z(1 + h). \qquad (5.4a, b, c)$$

At the same time we introduce a new independent variable ξ by

$$\xi = \ln(z_0/z), \quad z = z_0 e^{-\xi}, \qquad (5.5)$$

where z_0 is a reference value of z which is at our disposal. This variable ξ approaches $+\infty$ as z approaches zero. Indicating differentiation with respect to ξ by the symbol °, we obtain in place of (2.17) the system

$$h - \mathring{h} = \theta + \phi, \qquad (5.6)$$

$$(1 + h)\mathring{\theta} + 2\theta + \theta^2 + \tfrac{1}{4}(\beta_0'^2 - \gamma_0'^2) z^2 (1 + h)^2 = 0, \qquad (5.7)$$

$$(1 + h)\mathring{\phi} + \phi^2 + \tfrac{1}{4}(\beta_0'^2 - \gamma_0'^2) z^2 (1 + h)^2 = 0. \qquad (5.8)$$

We now formally expand h, θ, and ϕ as

$$h = h_0(\xi) + \tfrac{1}{2}z^2 h_1(\xi) + o(z^3), \tag{5.9a}$$

$$\theta = \tfrac{1}{2}z^2\theta_1(\xi) + o(z^3), \tag{5.9b}$$

$$\phi = \phi_0(\xi) + \tfrac{1}{2}z^2\phi_1(\xi) + o(z^3), \tag{5.9c}$$

recognizing that a certain amount of non-uniqueness is inherent in such an expansion. We remove the non-uniqueness by formally equating like powers of z^2 in the equation system, and by excluding solutions for h_0, h_1, θ_1, etc., which are exponential in ξ. This approach leads to the equations

$$h_0 - \overset{\circ}{h}_0 = \phi_0, \tag{5.10a}$$

$$3h_1 - \overset{\circ}{h}_1 = \theta_1 + \phi_1, \tag{5.10b}$$

$$(1+h_0)\,\overset{\circ}{\theta}_1 - 2h_0\theta_1 + \tfrac{1}{2}(\beta_0'^2 - \gamma_0'^2)(1+h_0)^2 = 0, \tag{5.10c}$$

$$(1+h_0)\,\overset{\circ}{\phi}_0 + \phi_0^2 = 0, \tag{5.10d}$$

$$(1+h_0)(\overset{\circ}{\phi}_1 - 2\phi_1) + h_1\overset{\circ}{\phi}_0 + 2\phi_0\phi_1 + \tfrac{1}{2}(\beta_0'^2 - \gamma_0'^2)(1+h_0)^2 = 0. \tag{5.10e}$$

The equation system can obviously be continued. We distinguish two cases:

Case (i). The functions ϕ_0 and h_0 are identically zero. If $\beta_0'^2 = \gamma_0'^2$, $\phi \equiv 0$, and $\theta_1 = k^2$, the solution (5.1) appears. In general we have

$$\phi_1 = \tfrac{1}{4}(\beta_0'^2 - \gamma_0'^2), \tag{5.11a}$$

$$\theta_1 = k^2 - \tfrac{1}{2}(\beta_0'^2 - \gamma_0'^2)\,\xi, \tag{5.11b}$$

$$h_1 = \tfrac{1}{3}k^2 + \tfrac{1}{36}(\beta_0'^2 - \gamma_0'^2)(1 - 6\xi). \tag{5.11c}$$

Here one degree of freedom lies in the choice of k^2 and of z_0. This solution is a weakly non-analytic one which corresponds to (5.1). In this case the secondary solution is not very different from that of (5.2), and the result (5.3) for the stagnation streamline applies approximately.

Case (ii). A non-zero solution to (5.10*a*) and (5.10*d*) is obtained. In such a solution, the freedom of choice of z_0 may be used to give the required degree of freedom. We may choose the solution with asymptotic form

$$h_0 = \frac{1}{\xi} + \frac{\ln\xi}{\xi^2} + \frac{(\ln\xi)^2 - \ln\xi + 2}{\xi^3} + O\left(\frac{(\ln\xi)^3}{\xi^4}\right), \tag{5.12a}$$

$$\phi_0 = \frac{1}{\xi} + \frac{\ln\xi + 1}{\xi^2} + \frac{(\ln\xi)^2 + \ln\xi + 1}{\xi^3} + O\left(\frac{(\ln\xi)^3}{\xi^4}\right). \tag{5.12b}$$

The corresponding θ_1 then has the asymptotic form

$$\begin{aligned}\theta_1 = {} & k^2[\xi^2 - 2\xi\ln\xi + (\ln\xi)^2 + 2\ln\xi - 2 + O(\xi^{-1}(\ln\xi)^3)] \\ & + \tfrac{1}{2}(\beta_0'^2 - \gamma_0'^2)[\xi - \ln\xi + 1 + O(\xi^{-1}(\ln\xi)^2)],\end{aligned} \tag{5.13}$$

where k^2 is an arbitrary constant.

For the secondary solution in case (ii) it is again convenient to expand F and G in a manner analogous to (5.9). We set

$$F = F_0(\xi) + \tfrac{1}{2}z^2F_1(\xi) + o(z^3), \tag{5.14a}$$

$$G = zG_0(z) + o(z^2), \tag{5.14b}$$

and obtain for the secondary equations

$$(1+h_0)\mathring{F}_0+\phi_0 F_0 = -M, \tag{5.15a}$$

$$(1+h_0)\mathring{G}_0-h_0 G_0+\tfrac{1}{2}(\beta_0'+\gamma_0')(1+h_0)F_0 = 0, \tag{5.15b}$$

$$(1+h_0)\mathring{F}_1-2(1+h_0)F_1+\phi_0 F_1+\phi_1 F_0+(\beta_0'-\gamma_0')(1+h_0)G_0 = 0. \tag{5.15c}$$

The solution takes the asymptotic form

$$F_0 = \tfrac{1}{2}M[-\xi+\ln\xi+1-\xi^{-1}\ln\xi+O(\xi^{-2}(\ln\xi)^2)]+f_0\phi_0, \tag{5.16}$$

$$\begin{aligned} G_0 = {} & \tfrac{1}{4}(\beta_0'+\gamma_0')M[\xi^2-\xi\ln\xi+\ln\xi-2+O(\xi^{-1}(\ln\xi)^2)] \\ & +\tfrac{1}{2}(\beta_0'+\gamma_0')f_0[1+\xi^{-1}+O(\xi^{-2}(\ln\xi))] \\ & +g_0[\xi-\ln\xi+\xi^{-1}(\ln\xi-1)+O(\xi^{-2}(\ln\xi)^2)]. \end{aligned} \tag{5.17}$$

In case (ii) M must be zero for finite velocity on the wall, and hence for the existence of a stagnation point. If $f_0 \neq 0$ a translation of the origin in the x direction can be used to make $f_0 = 0$. Thus we may set $f_0 = 0$ without loss of generality. The lowest-order solution for F may then be obtained from (5.15c). A stagnation streamline enters the origin in this case, with (5.4a) holding and with

$$y \doteq -\tfrac{1}{4}g_0 z[\ln(z_0/z)+\tfrac{1}{2}] \tag{5.18}$$

in place of (5.4b). This stagnation streamline is tangent to the body at the stagnation point. Other points along the x-axis are stagnation points into which no stagnation streamline enters. The author is unaware of any previous report of such points in inviscid flow.

6. Boundary-layer equations

The boundary-layer equations may be obtained directly from the viscous equations (2.5). Two essentially equivalent approaches are available: with the parameter a fixed we may carry out a transformation to the new variable $R^{\frac{1}{2}}z$, keeping the quantities H', α, β, and γ invariant; alternatively we may simply define the arbitrary parameter a so that R as defined in (2.5g) is unity. With a true boundary layer, outer boundary conditions for the boundary-layer equations correspond to inner boundary conditions for some given inviscid flow, in a limiting process in which $R \to \infty$.

The inner boundary conditions for the boundary-layer equations are those of zero slip

$$H(0) = 0, \quad H'(0) = 0, \quad \alpha(0) = 0, \quad \beta(0) = 0, \quad \gamma(0) = 0, \quad F(0) = 0, \quad G(0) = 0. \tag{6.1a–g}$$

The principal outer boundary conditions are

$$H'(\infty) = 1, \quad \alpha(\infty) = \alpha_0. \tag{6.2a, b}$$

The transformed quantities β_0' and γ_0' are of order $R^{-\frac{1}{2}}$ and approach zero in the limiting process. Thus we set

$$\beta'(\infty) = 0, \quad \gamma'(\infty) = 0, \tag{6.3a, b}$$

and conclude that β and γ are identically zero in the boundary-layer equations.

In the case $\alpha_0 = 0$ we conclude similarly that $\alpha'(\infty)$ (and $H''(\infty)$) is to be set equal to zero. In the case $1 > \alpha_0 > 0$ we analogously set the transformed values of h_0 and h_1 in (4.5) equal to zero for the external inviscid flow, provided we do not have $0 \leqslant 1-\alpha_0 \ll 1$.

With $1 > \alpha_0 \geqslant 0$, then, the boundary-layer equations become

$$H''' + HH'' + \tfrac{1}{2}(1+\alpha_0^2 - H'^2 - \alpha^2) = 0, \tag{6.4a}$$

$$\alpha'' + H\alpha' + \alpha_0 - H'\alpha = 0, \tag{6.4b}$$

for the primary flow, and

$$F'' + HF' - \tfrac{1}{2}(H'-\alpha)F = 0, \tag{6.5a}$$

$$G'' + HG' - \tfrac{1}{2}(H'+\alpha)G = 0, \tag{6.5b}$$

for the secondary flow. If $\alpha_0 = 0$, then α is identically zero and (6.4*a*) is the classical Falkner–Skan equation for an axisymmetric stagnation point. If $1 > \alpha_0 > 0$, then (6.4) are equivalent to those of Howarth (1951) for the boundary layer on a general stagnation point.

In the case $\alpha_0 = 1$, the quantities k, ϕ_1, and θ_1 approach zero in the boundary-layer limiting process. Thus in either the analytic case or case (i) of the last section we are led to the result

$$\alpha = H', \tag{6.6}$$

whereby (6.4*a*) and (6.4*b*) are the same equation, the same as the classical Falkner–Skan equation for a planar stagnation point. The right-hand side of (6.5*a*) is not zero in general in this case. Note that in this case (6.5*b*) is formally the homogeneous form of (6.4*b*); the solution has been given by Stuart (1959).

The case $0 \leqslant 1-\alpha_0 \ll 1$

In this case, although the non-analytic parts of the external inviscid solutions do approach zero in the limit $R \to \infty$ they do so very slowly. To represent the boundary layer properly we should permit a corresponding weak parametric dependence upon the Reynolds number. We define ϕ as in (5.4*b*) as $\frac{1}{2}(H'-\alpha)$, and write the boundary-layer equations

$$H''' + HH'' + \tfrac{1}{2}(1+\alpha_0^2) - H'^2 + 2H'\phi - 2\phi^2 = 0, \tag{6.7a}$$

$$\phi'' + H\phi' + \tfrac{1}{4}(1-\alpha_0)^2 - \phi^2 = 0. \tag{6.7b}$$

The outer boundary conditions are obtained if $\alpha_0 \neq 1$ from (4.5) by replacing z by $R^{-\frac{1}{2}}z$. This gives

$$\lim_{z\to\infty} R^{\frac{1}{2}(1-\alpha_0)} z^{-(1-\alpha_0)} (\phi - \tfrac{1}{2} + \tfrac{1}{2}\alpha_0) = h_0, \tag{6.8a}$$

$$\lim_{z\to\infty} R^{\frac{1}{2}(1-\alpha_0)} z^{-(1-\alpha_0)} (H'-1) = h_0. \tag{6.8b}$$

If $\alpha_0 = 1$ in case (ii) we obtain, with the external parameter z_0 unchanged,

$$\lim_{z\to\infty} \phi \ln(R^{\frac{1}{2}}z_0/z) = 1, \quad \lim_{z\to\infty} (H'-1)\ln(R^{\frac{1}{2}}z_0/z) = 1. \tag{6.9a, b}$$

Case (i) reduces to (6.6) above, with $\phi \equiv 0$.

The secondary equations corresponding to (6.7) are

$$F'' + HF' - \phi F = M, \quad G'' + HG' - (H'-\phi)G = 0, \tag{6.10a, b}$$

with $M = 0$ if $\alpha_0 \neq 1$. If $\alpha_0 = 1$, the homogeneous solution $F = \phi$ to (6.10*a*) exists, and this fact permits the general solution of (6.10*a*) by quadratures. In both (6.5) and (6.10), outer boundary conditions for the secondary solution will have a Reynolds-number dependence.

7. Conclusions

In examining the nature of the inviscid solutions here investigated, we are led to the conclusion that the flow patterns characteristic of irrotational stagnation-point flows are very special indeed. Although vorticity normal to the wall has little significant effect, any non-zero lateral vorticity approaching the wall is strongly amplified and changes the nature of the stagnation point in an essential manner. The velocity distributions corresponding to this amplified vorticity are non-analytic.

In rotational flow in general, a stagnation point of saddle-point type ($\alpha_0 > 1$) simply does not exist. In almost-planar flow ($\alpha_0 = 1$) also, stagnation points do not exist in general; cross-vorticity leads to infinite velocity on the wall, in agreement with the qualitative conclusion of Kronauer (1952). The case $\alpha_0 = 0$ also yields an example of stagnation points on a wall which have no entering streamlines. In the remaining cases, the stagnation point streamline has a curvature which is infinite at the stagnation point ($1 > \alpha_0 > 0$) or is finite there ($\alpha_0 = 0$) in a general rotational flow; in these cases the stagnation-point streamline is tangent to the wall at the stagnation point.

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